AP® CALCULUS AB/CALCULUS BC
2015 SCORING GUIDELINES

Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

(c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t)\,dt = 76.570$

$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$

Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

$2: \begin{cases} 1: \text{considers } R(3) \text{ and } D(3) \\ 1: \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t, $0 \le t \le 8$, is

$30 + \int_0^t [R(x) - D(x)]\,dx.$

$R(t) - D(t) = 0 \Rightarrow t = 0,\ 3.271658$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

$3: \begin{cases} 1: \text{considers } R(t) - D(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(d) $30 + \int_0^w [R(t) - D(t)]\,dt = 50$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{equation} \end{cases}$

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially ~~blocked, allowing water to drain~~ out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

$$\int_0^8 R(t)\,dt = \int_0^8 20\sin\frac{t^2}{35}\,dt = \boxed{76.570\ ft^3}$$

Durring the eight hour intervol, about 76.570 cubic feet of water flow into the drainpipe

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

Total water: $T(x)$ $\quad T'(x) = R(x) - D(x) = 20\sin\frac{t^2}{35} + .04t^3 - .4t^2 - .96t$

$T'(3) = 20\sin 9/35 + .04(27) - .4(9) - .96(3) = -.314 < 0$

after three hours, the amount of water in the pipe is **decreasing** because the derivative of the amount of water (the difference between water entering and leaving) is less than zero at 3 hours.

Continue problem 1 on page 5.

(c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

$T'(t) = 0$ @ $t = 0,\ 3.2716584$

$T(t) = T(0) + \int_0^t T'(t)\,dt = 30 + \int_0^t T'(t)\,dt$

$T(0) = 30$

$T(3.272) = 27.965$

$T(8) = 48.544$

after testing all critical numbers and endpoints for their values, the amount of water in the pipe achieves a minimum value of about 27.965 after about $\boxed{3.272 \text{ hours}}$

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

$50 = T(w)$

$50 = 30 + \int_0^w T'(t)\,dt$

$\boxed{20 = \int_0^w T'(t)\,dt} = \int_0^w [R(t) - D(t)]\,dt$

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1 & 2
1B

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

$$\int_0^8 20 \sin\left(\frac{t^2}{35}\right) dt = 76.57035295 \text{ ft}^3$$

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

At $t = 3$ hours, $R(t) < D(t)$ $\therefore$ The amount of water in the pipe is decreasing

$$R(3) = 20 \sin\left(\frac{3^2}{35}\right) = 5.086 \text{ ft}^3/\text{hour}$$

$$D(3) = -.04(3)^3 + .4(3)^2 + 0.96(3) = 5.4 \text{ ft}^3/\text{hour}$$

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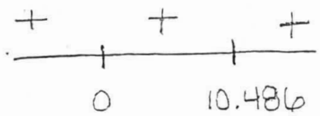
Continue problem 1 on page 5.

(c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

$R(t) = 0 = 20 \sin\left(\frac{t^2}{35}\right)$ @ $t = 0$, 10.486



abs minimum @ $t = 0$

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

$50 = 30 + \int_0^w 20 \sin\left(\frac{t^2}{35}\right) dt - \int_0^w -.04t^3 + .4t^2 + .96t \, dt$

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1 of 2
1C

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

$(0, 30)$

$R(8) = 20\sin\left(\frac{8^2}{35}\right)$

$= 20\sin\left(\frac{64}{35}\right)$

$\boxed{R(8) = 19.3392}$

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

$R(3) = 20\sin\left(\frac{3^2}{35}\right)$

$= 20\sin\left(\frac{9}{35}\right)$

$R(3) = 5.08637$

$D(t) = -0.04(3)^3 + 0.4(3)^2 + 0.96(3)$

$D(t) = 5.4$

The amount of water in the pipe is decreasing at time $t = 3$ hours, because the rate at which the water is draining out the other end of the pipe is greater, $D(3) = 5.4$, than the rate at which the rainwater flows into the drainpipe, $R(3) = 5.08637$.

Continue problem 1 on page 5.

2 of 2
1C

(c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

$$20\sin\left(\frac{t^2}{35}\right) = -0.04t^3 + 0.4t^2 + 0.96t$$

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

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Question 1

Overview

In this problem students were given $R(t)$, the rate of flow of rainwater into a drainpipe, in cubic feet per hour, and $D(t)$, the rate of flow of water out of the pipe, in cubic feet per hour. Both $R(t)$ and $D(t)$ are defined on the time interval $0 \le t \le 8$. The amount of water in the pipe at time $t = 0$ is also given. In part (a) students needed to use the definite integral to compute the amount of rainwater that flows into the pipe during the interval $0 \le t \le 8$. Students had to set up the definite integral $\int_0^8 R(t)\,dt$ and evaluate the integral using the calculator. In part (b) students should have recognized that the rate of change of the amount of water in the pipe at time t is given by $R(t) - D(t)$. Students were expected to calculate $R(3) - D(3)$ using the calculator and find that the result is negative. Therefore, the amount of water in the pipe is decreasing at time $t = 3$. In part (c) students had to find the time t, $0 \le t \le 8$, at which the amount of water in the pipe is at a minimum. Students were expected to set up an integral expression such as $30 + \int_0^t [R(x) - D(x)]\,dx$ for the amount of water in the pipe at time t. Students should have realized that an absolute minimum exists since they are working with a continuous function on a closed interval, and this minimum must occur at either a critical point or at an endpoint of the interval. Students were expected to use the calculator to solve $R(t) - D(t) = 0$ and find the single critical point at $t = 3.272$ on the interval $0 < t < 8$. Students should have stored the full value for t in the calculator and used the calculator to evaluate the function at the critical point and the endpoints. In this case the amount of water is at a minimum at the single critical point. In part (d) students were asked to write an equation involving one or more integrals that gives the time w when the pipe will begin to overflow. Students were expected to set up an equation using the initial condition, an integral expression, and the holding capacity of the pipe, such as
$30 + \int_0^w [R(t) - D(t)]\,dt = 50.$

Sample: 1A
Score: 9

The response earned all 9 points.

Sample: 1B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In parts (a) and (b), the student's work is correct. In part (c) the student works with $R(t)$ rather than $R(t) - D(t)$. In part (d) the student's work is correct.

Sample: 1C
Score: 3

The response earned 3 points: no points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student finds the rate at which water enters the pipe rather than the total amount. In part (b) the student's work is correct. In part (c) the student earned the first point for considering $R(t) - D(t) = 0$.

Question 2

At time $t \geq 0,$ a particle moving along a curve in the xy-plane has position $(x(t),\ y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right),\ e^{0.5t}\right).$ At $t = 1,$ the particle is at the point $(3, 5).$

(a) Find the x-coordinate of the position of the particle at time $t = 2.$

(b) For $0 < t < 1,$ there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

(c) Find the time at which the speed of the particle is 3.

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1.$

(a) $x(2) = 3 + \int_1^2 \cos\left(t^2\right)\, dt = 2.557$ (or 2.556)

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^{0.5t}}{\cos\left(t^2\right)}$

$\dfrac{e^{0.5t}}{\cos\left(t^2\right)} = 2$

$t = 0.840$

$2 : \begin{cases} 1 : \text{slope in terms of } t \\ 1 : \text{answer} \end{cases}$

(c) Speed $= \sqrt{\cos^2\left(t^2\right) + e^t}$

$\sqrt{\cos^2\left(t^2\right) + e^t} = 3$

$t = 2.196$ (or 2.195)

$2 : \begin{cases} 1 : \text{speed in terms of } t \\ 1 : \text{answer} \end{cases}$

(d) Distance $= \int_0^1 \sqrt{\cos^2\left(t^2\right) + e^t}\, dt = 1.595$ (or 1.594)

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

$\frac{dx}{dt}$ $\frac{dy}{dt}$

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

$$x(2) = 3 + \int_1^2 \cos(t^2)\,dt = 2.557$$

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

$$\frac{dy}{dt} = e^{0.5t}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)} = 2$$

$$\frac{dx}{dt} = \cos(t^2)$$

$t = .840$, $\cancel{-2.780}$ not in interval $0 < t < 1$

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Continue problem 2 on page 7.

(c) Find the time at which the speed of the particle is 3.

$$\|v\| = \sqrt{[\cos(t^2)]^2 + [e^{0.5t}]^2} = 3$$

$$t = 2.196$$

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

$$\text{Arc Length} = \int_0^1 \sqrt{[\cos(t^2)]^2 + [e^{0.5t}]^2}\, dt$$

$$= 1.595$$

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2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

$$3 + \int_1^2 \cos(t^2)\,dt = 4$$

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

$$\frac{y'(t)}{x'(t)} = 2 = \cancel{\frac{0.5e^{0.5t}}{-2t\sin t^2}} = \frac{e^{0.5t}}{\cos(t^2)}$$

$$\cancel{-4t\sin t^2 = 0.5e^{0.5t}} \qquad 2\cos t^2 = e^{0.5t}$$

$$t = 0.84$$

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Continue problem 2 on page 7.

(c) Find the time at which the speed of the particle is 3

$\sqrt{(\cos(t^2))^2+(e^{0.5t})^2}=3$ $\cos^2(t^2)+e^t=9$

$t=2.19$

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

$\int_0^1 \sqrt{x'(t)^2+y'(t)^2}\,dt$ $\int_0^1 \sqrt{\cos^2(t^2)+e^t}\,dt$

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2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

position functions $\left(\frac{1}{2t}\sin(t^2) + 5.158, 2e^{.5t}\right)$

$x = \frac{1}{2t}\sin(t^2) + C$

$\left(\frac{1}{2(2)}\sin(2^2) + 5.158, 2e^{.5(2)}\right)$

$3 = \frac{1}{2}\sin(1) + C$

$\left(\frac{1}{4}\sin(4) + 5.158\right)$

$6 - .8415 = C$

$C = 5.158$

$\boxed{4.969}$

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

$\frac{dy}{dx} = \frac{e^{.5t}}{\cos(t^2)} = \frac{2}{1}$

$t = 4.159$

$\cancel{\frac{1}{2}e^{.5t} = \cos(t^2)}$

$\cancel{\cos^{-1}\left(\frac{1}{2}e^{.5t}\right) = t^2}$

Continue problem 2 on page 7.

(c) Find the time at which the speed of the particle is 3.

$$3 = \sqrt{(x'(t))^2 + (y'(t))^2}$$

$$3 = \sqrt{(\cos(t^2))^2 + (e^{.5t})^2}$$

$$t = 2.196$$

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

$$\int_0^1 \left| \frac{e^{.5t}}{\cos(t^2)} \right| dt = 1.501$$

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AP® CALCULUS BC
2015 SCORING COMMENTARY

Question 2

Overview

In this problem students were given the velocity vector of a particle moving in the xy-plane with position $(x(t),\ y(t))$. The particle is at the point $(3, 5)$ at time $t = 1$. In part (a) students had to find the x-coordinate of the position of the particle at time $t = 2$. The x-coordinate of the position of the particle at $t = 1$ added to the net change from $t = 1$ to $t = 2$ produces the x-coordinate at $t = 2$, which is $x(1) + \int_1^2 \cos\left(t^2\right) dt$. Students were expected to evaluate this expression with the calculator. In part (b) students were given that there is a point on the curve at which the line tangent to the curve has a slope of 2. Students needed to find the time at which the particle was at that point. Students had to realize that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and then solve the equation $\frac{dy}{dx} = 2$ using the calculator. In part (c) students were asked to find the time at which the speed of the particle is 3. Students needed to solve the equation $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3$ using the calculator. In part (d) students were asked to find the total distance traveled by the particle from time $t = 0$ to time $t = 1$. Students needed to set up the integral expression $\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ and then evaluate this expression using the calculator.

Sample: 2A
Score: 9

The response earned all 9 points.

Sample: 2B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the student presents a definite integral equal to the change in the x-coordinate of the particle from time $t = 1$ to $t = 2$, so the first point was earned. The student presents the correct use of the initial condition, so the second point was earned. The student does not state the correct answer. In part (b) the student's work is correct. In part (c) the student presents an expression for the speed in terms of t set equal to the given speed of 3, so the first point was earned. The student does not state the correct answer to three decimal places, so the second point was not earned. In part (d) the student presents a definite integral equal to the distance the particle traveled from time $t = 0$ to $t = 1$, so the first point was earned. The student does not state an answer.

Sample: 2C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In part (a) the student does not present an integral, so the first point was not earned. The student attempts to anti-differentiate $\cos\left(t^2\right)$. Neither the second nor third points were earned. In part (b) the student presents a correct expression for slope in terms of t, so the first point was earned. The student does not state the correct answer. In part (c) the student's work is correct. In part (d) the student presents a definite integral whose integrand does not represent the speed of the particle.

Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

(a) $v'(16) \approx \dfrac{240 - 200}{20 - 12} = 5$ meters/min^2	1 : approximation
(b) $\int_0^{40} \lvert v(t)\rvert\, dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \le t \le 40$ minutes. $\int_0^{40} \lvert v(t)\rvert\, dt \approx 12 \cdot \lvert v(12)\rvert + 8 \cdot \lvert v(20)\rvert + 4 \cdot \lvert v(24)\rvert + 16 \cdot \lvert v(40)\rvert$ $= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150$ $= 2400 + 1920 + 880 + 2400$ $= 7600$ meters	$3 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$
(c) Bob's acceleration is $B'(t) = 3t^2 - 12t$. $B'(5) = 3(25) - 12(5) = 15$ meters/min^2	$2 : \begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$
(d) Avg vel $= \dfrac{1}{10}\int_0^{10} \left(t^3 - 6t^2 + 300\right) dt$ $= \dfrac{1}{10}\left[\dfrac{t^4}{4} - 2t^3 + 300t\right]_0^{10}$ $= \dfrac{1}{10}\left[\dfrac{10000}{4} - 2000 + 3000\right] = 350$ meters/min	$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

1 of 2

3A

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

a) $v'(16) \approx \dfrac{v(20) - v(12)}{20 - 12} = \dfrac{240 - 200}{8} = \dfrac{40}{8} = \dfrac{20}{4} = 5 \dfrac{m}{min^2}$

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

b) $\int_0^{40} |v(t)|dt$ represents the total distance in meters Johanna traveled between times $t=0$ and $t=40$ minutes.

$\int_0^{40} |v(t)|dt \approx [12(200) + 8(240) + 4(220) + 16(150)]$

$= 2400 + 1920 + 880 + 2400$

$= 7600$ meters

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Continue problem 3 on page 15.

2 & 2
3A 2

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

c) $\text{acceleration} = B'(t) = 3t^2 - 12t$

$B'(5) = 3(5)^2 - 12(5)$
$= 75 - 60 = 15 \frac{m}{min^2}$

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

d) $\text{Avg. velocity} = \frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300)\,dt$

$= \frac{1}{10} \cdot \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10}$

$= \frac{1}{10} \cdot \left[\frac{10000}{4} - 2000 + 3000 \right]$

$= \frac{1}{10}[3500] = 350 \frac{m}{min}$

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NO CALCULATOR ALLOWED

3B1

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

$$v'(16) = \frac{v(20) - v(12)}{20 - 12} = \frac{40}{8} = 5$$

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(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

$\int_0^{40} |v(t)|\, dt$ is the total distance covered by Johanna from $t = 0$ to $t = 40$.

$200(12) + 240(8) + 220(4) + 150(16)$

$= 2400 + 1920 + 880 + 2400$

$= 4800 + 2800 = 7600$

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Continue problem 3 on page 15.

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3B2

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

$B'(t) = 3t^2 - 12t$

$B'(5) = 3 \cdot 25 - 12 \cdot 5$

$= 75 - 60 = 15$ (m/minute)/minute

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

$\frac{1}{10} \int_0^{10} t^3 - 6t^2 + 300 \, dt$ meters/minute

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1 of 2

3C1

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12 8 4 6

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

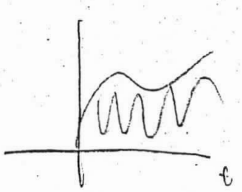
3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

$$v'(16) \approx \frac{240-200}{20-12} = \frac{40}{6} = \boxed{\frac{20}{3}}$$

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

$\int_0^{40} |v(t)|$ is the total distance travelled (including when Johanna jogs backwards as positive distance)



positive

$$\text{total distance} = (12)(200) + 8(240) + 4(+220) + 6(150)$$

Continue problem 3 on page 15.

2 of 2

3C2

NO CALCULATOR ALLOWED

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

$$a = \frac{dv}{dt} = 3t^2 - 12t$$

$$a(5) = 3(25) - 12(5)$$

$$75 - 60$$

$$= \boxed{15 \text{ meters}/\text{minute}^2}$$

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

$a = \frac{dv}{dt}$

$v = \int a \, dt$

$$\text{average velocity} = \frac{\int_0^{10} 3t^2 - 12t}{10 - 0}$$

$$= \left[\frac{t^3 - 6t^2}{10}\right]_0^{10} = \frac{10^3 - 6(10^2)}{10} - 0$$

$$= \frac{1000 - 600}{10}$$

$$= \frac{400}{10} = \boxed{40 \text{ meters}/\text{minute}}$$

GO ON TO THE NEXT PAGE.

Question 3

Overview

In this problem students were given a table of values of a differentiable function v, the velocity of a jogger, in meters per minute, jogging along a straight path for selected values of t in the interval $0 \le t \le 40$. In part (a) students were expected to know that $v'(16)$ can be estimated by the difference quotient $\frac{v(20) - v(12)}{20 - 12}$. In part (b) students were expected to explain that the definite integral $\int_0^{40} |v(t)|\, dt$ gives the total distance jogged, in meters, by Johanna over the time interval $0 \le t \le 40$. Students had to approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the subintervals $[0, 12]$, $[12, 20]$, $[20, 24]$, $[24, 40]$, and values from the table. In part (c) students were given a cubic function B, the velocity of a bicyclist, in meters per minute, riding along the same straight path used by Johanna for $0 \le t \le 10$. Students should have known that $B'(t)$ gives Bob's acceleration at time t. Students were expected to find $B'(t)$ using derivatives of basic functions and then evaluate $B'(5)$. In part (d) students had to set up the definite integral $\frac{1}{10}\int_0^{10} B(t)\, dt$ that gives Bob's average velocity during the interval $0 \le t \le 10$. Students needed to evaluate this integral using basic antidifferentiation and the Fundamental Theorem of Calculus.

Sample: 3A
Score: 9

The response earned all 9 points.

Sample: 3B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student's work is correct. In part (b) the student does not include "meters," so the explanation point was not earned. The student's Riemann sum and approximation are correct. In part (c) the student's work is correct. In part (d) the student's integral is correct.

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In part (a) the student attempts to simplify the correct difference quotient but makes an arithmetic error. In part (b) the student did not earn the explanation point because the time interval and the distance units (meters) are not included. The right Riemann sum has exactly one error. The student earned the point because 7 out of the 8 components are correct. The student did not earn the approximation point as a result of an error in the Riemann sum. In part (c) the student's work is correct. In part (d) the student uses $B'(t)$ in the integral instead of $B(t)$.

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Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

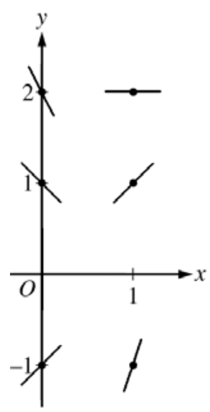
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

(a)



$2 : \begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

$2 : \begin{cases} 1 : \text{considers } \left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$2x - y = m$

$2x - (mx + b) = m$

$(2 - m)x - (m + b) = 0$

$2 - m = 0 \Rightarrow m = 2$

$b = -m \Rightarrow b = -2$

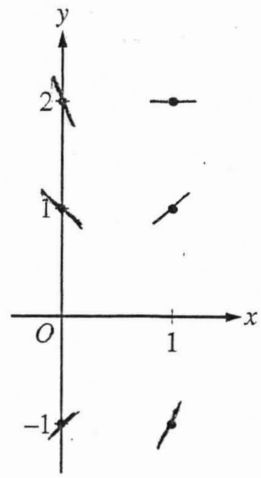
Therefore, $m = 2$ and $b = -2$.

$3 : \begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$$\frac{dy}{dx} = 2x - y$$

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y)$$

$$\boxed{\frac{d^2y}{dx^2} = 2 - 2x + y}$$

In Quadrant II, $x < 0$ and $y > 0$,

so $\frac{d^2y}{dx^2} = 2 - 2x + y > 0$,

Thus $\boxed{\text{all solution curves in Quadrant II are concave up.}}$

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Continue problem 4 on page 17.

4A2
2 of 2

NO CALCULATOR ALLOWED

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$$\frac{dy}{dx} = 2x - y = 2 \cdot 2 - 3 = 1$$

$\boxed{\text{Neither}}$, as $\frac{dy}{dx} \neq 0$ at $x = 2$.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y$, $y = mx + b$

$\frac{dy}{dx} = m = 2x - y$

$m = 2x - (mx + b)$

$m = (2-m)x - b$, equate coefficients

$2 - m = 0$

$m = 2$

$-b = m$

$b = -m = -2$.

$\boxed{m = 2, b = -2}$

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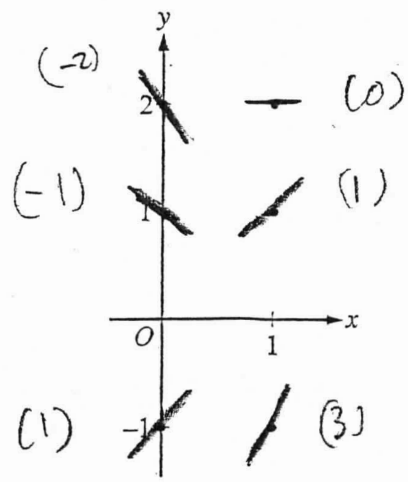
4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2 - 2x + y$$

-+ | ++
-- | +-

always negative   always positive

$$= 2 - 2(X) + y$$

$$= 2 + 2|x| + y = > 0$$

$\therefore$ concavity in Quadrant II is always concave up

Continue problem 4 on page 17.

4B2
2 of 2

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$$\frac{d^2y}{dx^2} = 2-2(2)+3 = 2-4+3 = -2+3 = 1 > 0$$

∴ concave up means minimum

∴ According to the second derivative test, at $x=2$ the second derivative is positive and ∴ there is a minimum at that point

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$m = 2x - y$

$y = (2x - y)x + b$

$y = 2x^2 - \cancel{yx} + b$

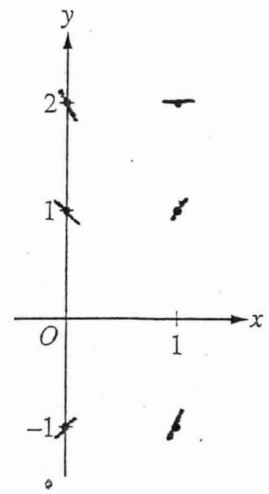
$y = 2x - \frac{dy}{d\cancel{x}}$

GO ON TO THE NEXT PAGE.

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

$y \backslash x$	0	1
-1	1	3
1	-1	1
2	-2	0



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$\frac{dy}{dx} = 2x - y =$

$\frac{d^2y}{dx^2} = 2 - y'$

$0 = 2 - y'$

$0 = 2 - (2x - y)$

$0 = 2 - 2x + y$

$y = 2x - 2$

$x = \frac{y+2}{2}$

$2x - 2 = 0$

$2x = 2$

$x = 1$

$x = \frac{y+2}{2}$

$0 = y + 2$

$y = 2$

concave down at $(1,2)$ because its a relative max.

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Continue problem 4 on page 17.

4C2

2 of 2

NO CALCULATOR ALLOWED

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$\frac{dy}{dx} = 2x - y$

$\int dy + y = \int 2x\, dx$

$y + y' = x^2$

$y' = x^2 - y$

$y' = (2)^2 - (3)$

$y' = 1$

f has a relative min at $x = 2$ because at $f(2)$ it is 3.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y$

$\int dy + y = \int 2x\, dx$

$y + y' = x^2 + c$

$y' = x^2 - y + c$

$y' = (2)^2 - 3 + c$

$y' = c$

$y - 3 = m(x - 2)$

$y = x + 1$

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Question 4

Overview

In this problem students were to consider the first-order differential equation $\frac{dy}{dx} = 2x - y$. In part (a) students were given an xy-plane with 6 labeled points and were expected to sketch a slope field by drawing a short line segment at each of the six points with slopes of $2x - y$. In part (b) students needed to use implicit differentiation and the fact that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ to obtain $\frac{d^2y}{dx^2} = 2 - 2x + y$. Students were expected to explain that for points in Quadrant II, $x < 0$ and $y > 0$ so $\frac{d^2y}{dx^2} > 0$. Thus, any solution curve for the differential equation that passes through a point (x, y) in Quadrant II must be concave up at (x, y). In part (c) students were asked to consider the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 3$. Students had to determine if $(2, 3)$ is the location of a relative minimum, a relative maximum, or neither for f and justify the answer. Students were expected to show that $\frac{dy}{dx} \neq 0$ at $(2, 3)$ and conclude that $(2, 3)$ is neither the location of a relative minimum nor a relative maximum. In part (d) students were asked to find the values of the constants m and b so that the linear function $y = mx + b$ satisfies the differential equation $\frac{dy}{dx} = 2x - y$. Students were expected to show that if $y = mx + b$, then $\frac{dy}{dx} = m$. Using a substitution in $\frac{dy}{dx} = 2x - y$ leads to $2x - y = m$ and thus $2x - (mx + b) = m$. This equation enabled the student to find the values of m and b.

Sample: 4A
Score: 9

The response earned all 9 points.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student's work is correct. In part (b) the student's work is correct, including correct reasoning about the sign of the second derivative using the signs of x and y in Quadrant II. In part (c) the student does not consider $\frac{dy}{dx}$, so the first point was not earned. The student considers $\frac{d^2y}{dx^2}$, which cannot be used as justification, and the student incorrectly identifies $x = 2$ as a minimum. The second point was not earned. In part (d) the student earned the first 2 points for declaring that $m = 2x - y$. In doing so, the student communicates that the derivative of the linear function is its slope m (the first point) and connects the differential equation and its linear solution by equating the derivatives (the second point). The student does not arrive at an answer.

Sample: 4C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for the correct second derivative in x and y, shown in the work where the student writes $0 = 2 - (2x - y)$. In part (c) the student incorrectly solves

Question 4 (continued)

the differential equation and then, from that work, finds an incorrect expression for the first derivative. The student is not eligible for any points. In part (d) the student attempts to solve the differential equation by separation of variables and uses the point $(2, 3)$, which is not relevant to the question asked.

Question 5

Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Find $\int f(x)\,dx$.

(a) $f(4) = \dfrac{1}{4^2 - 3\cdot 4} = \dfrac{1}{4}$ $\qquad f'(4) = \dfrac{3 - 2\cdot 4}{\left(4^2 - 3\cdot 4\right)^2} = -\dfrac{5}{16}$

An equation for the line tangent to the graph of f at the point whose x-coordinate is 4 is $y = -\dfrac{5}{16}(x - 4) + \dfrac{1}{4}$.

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

(b) $f'(x) = \dfrac{4 - 2x}{\left(x^2 - 4x\right)^2}$ $\qquad f'(2) = \dfrac{4 - 2\cdot 2}{\left(2^2 - 4\cdot 2\right)^2} = 0$

$f'(x)$ changes sign from positive to negative at $x = 2$.
Therefore, f has a relative maximum at $x = 2$.

$2 : \begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

(c) $f'(-5) = \dfrac{k - 2\cdot(-5)}{\left((-5)^2 - k\cdot(-5)\right)^2} = 0 \Rightarrow k = -10$

1 : answer

(d) $\dfrac{1}{x^2 - 6x} = \dfrac{1}{x(x - 6)} = \dfrac{A}{x} + \dfrac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$

$x = 0 \Rightarrow 1 = A\cdot(-6) \Rightarrow A = -\dfrac{1}{6}$

$x = 6 \Rightarrow 1 = B\cdot(6) \Rightarrow B = \dfrac{1}{6}$

$\dfrac{1}{x(x - 6)} = \dfrac{-1/6}{x} + \dfrac{1/6}{x - 6}$

$\int f(x)\,dx = \int\left(\dfrac{-1/6}{x} + \dfrac{1/6}{x - 6}\right)dx$

$= -\dfrac{1}{6}\ln|x| + \dfrac{1}{6}\ln|x - 6| + C = \dfrac{1}{6}\ln\left|\dfrac{x - 6}{x}\right| + C$

$4 : \begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

5A

1 of 2

5. Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}.$$

(a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

$$f'(4) = \frac{3 - 2(4)}{((4)^2 - 3(4))^2} = \frac{3-8}{(16-12)^2} = \frac{-5}{(4)^2} = -\frac{5}{16}$$

$$f(4) = \frac{1}{(4)^2 - 3(4)} = \frac{1}{16-12} = \frac{1}{4}$$

$$y - \frac{1}{4} = -\frac{5}{16}(x-4)$$

$$y = -\frac{5}{16}(x-4) + \frac{1}{4}$$

(b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

$$f'(2) = \frac{4 - 2(2)}{(4 - 4(2))^2} = \frac{0}{16} = 0$$

By 1st Derivative test,
there is a relative maximum
@ $x = 2$ because the derivative
value changes from positive to negative.

Continue problem 5 on page 19.

5A
2 of 2

(c) Find the value of k for which f has a critical point at $x = -5$.

$$f'(5) = 0 = \frac{k - 2(-5)}{(25 - k(-5))^2}$$

$$0 = \frac{k+10}{(25+5k)^2}$$

$$k = -10$$

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\int f(x)\,dx$.

$$f(x) = \frac{1}{x(x-6)}$$

$$\int f(x)dx = \int \frac{dx}{x(x-6)} = \int \frac{A}{x}dx + \int \frac{B\,dx}{x-6}$$

$1 = A(x-6) + Bx$ let $x = 0$

$\hookrightarrow 1 = -6A$ $A = -\frac{1}{6}$

let $x = 6$

$\hookrightarrow 1 = 6B$ $B = \frac{1}{6}$

$$\int f(x)dx = -\frac{1}{6}\int \frac{dx}{x} + \frac{1}{6}\int \frac{dx}{x-6}$$

$$\int f(x)dx = -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C$$

$$\int f(x)dx = \frac{1}{6}(\ln|x-6| - \ln|x|) + C$$

$$\int f(x)dx = \frac{1}{6}\ln\left|\frac{x-6}{x}\right| + C$$

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NO CALCULATOR ALLOWED

5B
1 of 2

5. Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}.$$

(a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

$$f'(x) = \frac{3-2x}{(x^2-3x)^2}$$

$x = 4$,

$$f(4) = \frac{1}{16-12} = \frac{1}{4}$$

$$f'(4) = \frac{3-8}{(16-12)^2} = \frac{5}{16}$$

$\therefore y - \frac{1}{4} = \frac{5}{16}(x-4)$

$y = \frac{5}{16}x - 1$

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(b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

$$f'(x) = \frac{4-2x}{(x^2-4x)^2}$$

$f'(2) = 0$

$$f''(x) = \frac{-2(x^2-4x)^2 - (4-2x)\cdot(x^2-4x)\cdot(2x-4)}{(x^2-4x)^4}$$

$f''(2) = -\frac{1}{2} < 0$

$\therefore f$ has a relative maximum at $x=2$

Continue problem 5 on page 19.

5B
2 of 2

(c) Find the value of k for which f has a critical point at $x = -5$.

$f'(5) = 0$

$f'(5) = \dfrac{k-10}{(25-5k)^2} = 0$

$k = 10$

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\int f(x)\,dx$.

$\int f(x)\,dx = \int \dfrac{1}{x^2-6x}\,dx$

$\dfrac{1}{x^2-6x} = \dfrac{1}{x(x-6)}$

$= \dfrac{A}{x} + \dfrac{B}{x-6}$

$\therefore A(x-6) + Bx = 1$

$\begin{cases} A + B = 0 \\ -6A = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{6} \\ B = \frac{1}{6} \end{cases}$

$\int f(x)\,dx = -\frac{1}{6}\int \frac{1}{x}\,dx + \frac{1}{6}\int \frac{1}{x-6}\,dx$

$= -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C$

$= \frac{1}{6}\ln\left|\frac{x-6}{x}\right| + C$

Do not write beyond this border.

5. Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}$.

(a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

$k=3$ $x=4$ $y=\frac{1}{4}$

$f(4) = \frac{1}{16-3(4)} = \frac{1}{16-12} = \frac{1}{4}$

$\frac{3-2(4)}{(4^2-3(4))^2} = \frac{-5}{(16-12)^2}$

$\boxed{y - \frac{1}{4} = -\frac{5}{16}(x-4)}$

$\frac{-5}{4^2} = -\frac{5}{16}$

(b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

$4-6$

$+ \quad | \quad -$

2

$f'(x) = \frac{4-2x}{(x^2-4(2))^2} = \frac{4-2x}{(x^2-8)^2}$

$a-8$

$\frac{4-2(2)}{(4-8)^2} = \frac{0}{(-4)^2} = \frac{0}{16} = 0$

maximum at $x=2$ because the graph increases up to $x=2$ and then decreases.

Do not write beyond this border.

Continue problem 5 on page 19.

(c) Find the value of k for which f has a critical point at $x = -5$.

$f'(-5) = \frac{k-2x}{(x^2-kx)^2} = 0$

$\frac{k+10}{(25+5k)^2} = 0$

$125 + 25k^2$

$k+10 \; (125^{-1}) +$

$\left(\frac{k-2(-5)}{25+5k}\right)^2$

(d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\int f(x)\,dx$.

$\int f(x)\,dx$

$u = x^2 - 6x$

$du = 2x - 6\,dx$

$dx = \frac{1}{2x-6}\,du$

$\int \frac{1}{x^2-6x}\,dx$

$\int (x^2-6x)^{-1}\,dx$

$u^{-1} \cdot \frac{1}{2x-6}\,du$

$u^0 \; \frac{1}{2x-6} + C$

$\boxed{\frac{1}{2x-6} + C}$

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Question 5

Overview

In this problem students were given $f(x) = \dfrac{1}{x^2 - kx}$ and $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$, where the parameter, k, is a nonzero constant. In part (a) for $k = 3$, students were asked to write an equation for the line tangent to the graph of f at the point with $x = 4$. Students needed to compute $f(4)$ and $f'(4)$ and then use those values to produce an equation. In part (b) for $k = 4$, students needed to determine whether f had a relative minimum, a relative maximum, or neither at $x = 2$. Students were expected to confirm that $f'(2) = 0$ and apply the First Derivative Test. Since f' changes sign from positive to negative at $x = 2$, students should have concluded that f has a relative maximum at $x = 2$. In part (c) students had to find the value of k for which f has a critical point at $x = -5$. Students were expected to solve $f'(-5) = 0$ to determine that $k = -10$. In part (d) students were expected to use partial fraction decomposition to rewrite $f(x)$ as a sum of rational expressions. The result is used to find $\int f(x)\,dx$. The partial fraction decomposition yields $\dfrac{1}{x(x-6)} = \dfrac{A}{x} + \dfrac{B}{x-6} = \dfrac{-1/6}{x} + \dfrac{1/6}{x-6}$, and the general antiderivative is $-\dfrac{1}{6}\ln|x| + \dfrac{1}{6}\ln|x-6| + C$.

Sample: 5A
Score: 9

The response earned all 9 points.

Sample: 5B
Score: 6

The response earned 6 points: 1 point in part (a), 1 point in part (b), no points in part (c), and 4 points in part (d). In part (a) the student uses f' to determine the slope but makes an arithmetic error, so the student did not earn the first point. The student uses the slope to present a line that passes through the given point, so the second point was earned. In part (b) the student attempts to use the Second Derivative Test. The student shows that $f'(2) = 0$, so the first point was earned. The student makes an error in the computation of $f''(2)$, so the second point was not earned. In part (c) the student considers $x = 5$ instead of $x = -5$. In part (d) the student's work is correct.

Sample: 5C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point. Because the student's justification does not refer to the sign of the first derivative of f, the student did not earn the second point. In part (c) the student does not report a value for k. In part (d) the student does not find a partial fraction decomposition, and the antiderivative is incorrect.

Question 6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

(a) Let a_n be the nth term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n\to\infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \frac{1}{3}$.

$$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$$

(b) The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$.

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

$$3 : \begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$$

(c) The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \cdots\right)$$
$$= x - \frac{1}{2}x^2 + 2x^3 + \cdots$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

$$3 : \begin{cases} 1 : \text{first four nonzero terms of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$$

NO CALCULATOR ALLOWED

6A
1 of 2

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

$$\lim_{n\to\infty} \left| \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} \right|$$

$$= \lim_{n\to\infty} |-3x| = \lim_{n\to\infty} |3x| < 1$$

$$|x| < \frac{1}{3}$$

$$R = \frac{1}{3}$$

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Continue problem 6 on page 21.

NO CALCULATOR ALLOWED

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

$f = x - \frac{3}{2}x^2 + 3x^3 - \frac{3^3}{4}x^4$

$f' = 1 - 3x + 9x^2 - 27x^3$

$a = 1 \quad r = -3x$

$f' = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$ for $|x| < \frac{1}{3}$

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$g(x) = e^x f(x) \approx (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})(x - \frac{3}{2}x^2 + 3x^3)$

$= x - \frac{3}{2}x^2 + 3x^3 + x^2 - \frac{3}{2}x^3 + \cancel{3x^4} + \frac{x^3}{2!} - \cancel{\frac{3}{4}x^4} + \cancel{\frac{3}{2}x^5}$

$-\frac{3}{2} + \frac{2}{2} = -\frac{1}{2}$

$= x - \frac{3}{2}x^2 + x^2 + 3x^3 - \frac{3}{2}x^3 + \frac{1}{2}x^3$

$= x - \frac{1}{2}x^2 + 2x^3$

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6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^{(n+1)-1}}{n+1} x^{n+1}}{\frac{(-3)^{n-1}}{n} x^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{(-3)^n}{(-3)^{n-1}} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| -3x \right| = |-3x| < 1$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

$$\boxed{R = \frac{1}{3}}$$

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Continue problem 6 on page 21.

NO CALCULATOR ALLOWED

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

$$f'(x) = \frac{d}{dx}\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{n}x^n = \sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{n}\cdot n x^{n-1} = \boxed{\sum_{n=1}^{\infty}(-3)^{n-1}x^{n-1}} = f'(x)$$

$$f'(x) \approx \boxed{1 + (-3)x + (-3)^2x^2 + (-3)^3x^3}$$

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(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

$$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n \approx \boxed{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$$

$$g(x) = e^x f(x)$$

$$g(0) = e^0 f(0) = f(0) = 0 \Rightarrow \boxed{g(0) = 0}$$

$$\sum_{n=0}^{\infty}\frac{g^n(c)}{n!}(x-c)^n$$

GO ON TO THE NEXT PAGE.

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

$$\frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = (-3)(1)x$$

$-3x < 1$

$0 < x < -\frac{1}{3}$

$x > 0$
$x < -\frac{1}{3}$

$x < \frac{1}{3}$

R is infinite?

Continue problem 6 on page 21.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

$$1 - 3x + 9x^2 - \frac{27x^3}{4}$$

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

$$1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$$

$$x - \frac{3}{2}x^3$$

Question 6

Overview

In this problem students were presented with the Maclaurin series $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ for a function f. The Maclaurin series converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series. In part (a) students were asked to use the ratio test to find R. Students were expected to evaluate $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ and use this limit to find R. Students were expected to show that $|x| < \frac{1}{3}$, and thus the radius of convergence is $R = \frac{1}{3}$. In part (b) students were asked to write the first four nonzero terms of the Maclaurin series for f', then express f' as a rational function for $|x| < R$. By using term-by-term differentiation, the first four nonzero terms are $1 - 3x + 9x^2 - 27x^3$. Because this series is geometric with a common ratio of $-3x$, the rational function is $f'(x) = \frac{1}{1+3x}$. In part (c) students needed to write the first four nonzero terms of the Maclaurin series for e^x and use this series to write a third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$. After showing that the first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, students were expected to multiply to determine that the third-degree Taylor polynomial desired is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

Sample: 6A
Score: 9

The response earned all 9 points.

Sample: 6B
Score: 6

The response earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student writes the correct first four nonzero terms of f', so the first 2 points were earned. There is no rational function presented. In part (c) the student writes the correct first four nonzero terms of the Maclaurin series for e^x, so the first point was earned. The student does not present the correct third-degree Taylor polynomial for g.

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student writes the correct setup, so the first point was earned. The student does not indicate a limit, so the second point was not earned. The student does not determine a radius of convergence, so the third point was not earned. In part (b) the student writes three of the correct first four nonzero terms of the Maclaurin series for f', so 1 of the first 2 points was earned. There is no rational function presented. In part (c) the student writes the correct first four nonzero terms of the Maclaurin series for e^x, so the first point was earned. The student does not present the correct third-degree Taylor polynomial for g.